esi Engineering Bulletin
Electro Scientific Industries
7524 S. W. MACADAM AVE. • PORTLAND 19, OREGON

COMPUTERS
no. 19
JANUARY 1960
REPLACES NOVEMBER 1959

EVALUATING RESIDUES AND COEFFICIENTS OF HIGH ORDER POLES

by

Dov Hazony and Jack Riley

Presented
on August 21, 1959 at the
Automatic Control Session of the
WESTERN ELECTRONIC SHOW AND CONVENTION

WESCON
IRE
WEMA

Litho in U.S.A.
©Electro Scientific Industries, Inc. 1960

SIMPLIFIED TECHNIQUE FOR EVALUATING RESIDUES IN THE PRESENCE OF HIGH ORDER POLES

by

Dov Hazony and Jack Riley

SUMMARY

The powerful Laplace transform method for transient analysis by the partial fraction expansion technique has become quite popular in the fields of circuit and servo design. The theory of residues is usually used to find the coefficients of these fractions. The process is quite simple until second and higher order poles are included in the denominator. Previously, this has required a return to the calculus to find the additional coefficients required.

This paper discloses a simple technique for finding these additional coefficients by algebraic processes. As a result both manual and machine computation can be performed more easily. The technique is described and its mathematical basis is rigorously proven.

I INTRODUCTION

This paper describes a simplified technique for the evaluation of the coefficients of partial fraction expansions. The method is based on the following relationships.

Rule: For a normalized ratio of polynomials:

A. If the denominator is one degree higher than the numerator, the sum of the residues is one.

B. If the denominator is two or more degrees higher than the numerator, the sum of the residues is zero.

The proof for these rules is given in the appendix.

Several additional rules are derived from these to permit algebraic evaluation of partial fraction coefficients in cases for which the degree of the numerator is equal or higher than that of the denominator.

The application of these rules provides a technique for finding one residue as a simple function of the others. By this means it is possible to find all of the coefficients of a partial fraction expansion by algebra even when higher order poles are involved. This technique reduces the effort of computation considerably whether the coefficients are being found directly, graphically with computational aids such as the Spirule or Nomoto's log - s plane charts, or with computers, either digital or algebraic such as the ESIAC. Before exploring this technique it will be useful to define the terms residue and normalized ratio as they are used in this paper.

Residue refers only to the coefficients of terms with first degree denominators in a partial fraction expansion. Constants, coefficients of positive powers of s, and coefficients of terms with higher degree denominators are referred to as coefficients only.

Normalization refers to the process of obtaining a coefficient of one for the highest power term in the polynomials of the numerator and denominator. This can always be accomplished by factoring out the ratio of the coefficients of the highest order terms before making the partial fraction expansion. This ratio is then used as a multiplier for all of the resulting coefficients.

II APPLICATIONS OF THE RULE

The usefulness of this relationship of the sum of the residues for practical partial fraction expansions can be easily demonstrated by several examples.

A. FIRST ORDER POLES ONLY:

$$\frac{(s+a)}{(s+b_1)(s+b_2)(s+b_3)} = \frac{A}{(s+b_1)} + \frac{B}{(s+b_2)} + \frac{C}{(s+b_3)}$$

The residues A and B are found by the usual technique of multiplying by $(s+b_i)$ and setting $s=-b_i$.

$$A = \frac{(a-b_1)}{(b_2-b_1)(b_3-b_1)} \qquad B = \frac{(a-b_2)}{(b_1-b_2)(b_3-b_2)}$$

then since $A+B+C=0$ the remaining residue C becomes

$$C = -(A+B)$$

Usually A and B are in numeric form so that C can be easily found. If there are only two poles and no zeros the residues will be equal in magnitude and opposite in sign. If there is only one less zero than pole the sum of the normalized residues will be one. If the degree of the numerator is equal to or higher than that of the denominator algebraic methods for finding the coefficients are still possible. Some of the techniques which can be used are given in section III.

B. ONE SECOND ORDER POLE:

$$\frac{1}{(s+b_1)^2(s+b_2)} = \frac{A}{(s+b_1)^2} + \frac{B}{(s+b_1)} + \frac{C}{(s+b_2)}$$

A and C can be found as above.

$$A = \frac{1}{(b_2-b_1)} \qquad C = \frac{1}{(b_1-b_2)^2}$$

By our definition only B and C are residues.
Since $B+C = 0$:

$$B = -C$$

Thus an answer is obtained quickly and simply by inspection instead of having to make a complicated differentiation.

The evaluation of coefficients in the presence of one second order pole can be generalized. Our problem can be stated as follows:

Given:

$$\frac{\prod_i (s+a_i)}{(s+b_0)^2 \prod_j (s+b_j)} = \frac{C}{(s+b_0)^2} + \frac{R_0}{(s+b_0)} + \sum_{k=1}^{n} \frac{R_k}{(s+b_k)}$$

$$i = 1, 2, \cdots, p$$
$$j = 1, 2, \cdots, n$$
$$p < (n+2)$$

Find: C, R_0, all R_k.

The values of R_k can be found as before and can be written as:

$$R_k = \left. \frac{(s+b_k) \prod_i (s+a_i)}{(s+b_0)^2 \prod_j (s+b_j)} \right|_{s=-b_k}$$

The value of C is given by:

$$C = \left. \frac{\cancel{(s+b_0)^2} \prod_i (s+a_i)}{\cancel{(s+b_0)^2} \prod_j (s+b_j)} \right|_{s=-b_0}$$

Since the sum of the residues must be equal to zero or one we find that

$$R_0 = \delta_{(p+1)(n+2)} - \sum_{k=1}^{n} R_k$$

where

$$\delta_{(p+1)(n+2)} = \begin{cases} 1 & \text{for } p+1 = n+2 \\ 0 & \text{for } p+1 \neq n+2 \end{cases}$$

C. ONE THIRD ORDER POLE:

From the equation:

$$\frac{(s+a)}{(s+b_1)^3(s+b_2)} = \frac{A}{(s+b_1)^3} + \frac{B}{(s+b_1)^2} + \frac{C}{(s+b_1)} + \frac{D}{(s+b_2)}$$

We find as before:

$$A = \frac{(a-b_1)}{(b_2-b_1)} \qquad D = \frac{(a-b_2)}{(b_1-b_2)^3}$$

then since $C+D=0$

$$C = -D$$

to find B both sides of the equation are multiplied by $(s+b_1)$ to obtain a new equation with only a second order pole. The new polynomial ratio is expanded into a partial fraction and the terms of this expansion are equated to similar ones from $(s+b_1)$ times the original expansion to give:

$$\frac{(s+a)}{(s+b_1)^2(s+b_2)} = \frac{A}{(s+b_1)^2} + \frac{B}{(s+b_1)} + \frac{F}{(s+b_2)}$$

Note: F could have been written

$$\frac{F}{s+b_2} = C + \frac{D(s+b_1)}{s+b_2}$$

but it is simpler to evaluate and use if it is taken from the new equation as indicated.

Both A and B are the same as in the above equation and F can be found easily:

$$F = \frac{a-b_2}{(b_1-b_2)^2}$$

then since $F+B=0$

$$B = -F$$

D. ONE HIGH ORDER POLE:

The same technique can be extended to include all such functions with a single high order pole.

The equation to be solved is:

$$\frac{\prod_i (s+a_i)}{(s+b_0)^m \prod_j (s+b_j)} = \sum_{\ell=1}^{m} \frac{C_\ell}{(s+b_0)^\ell} + \sum_{k=1}^{n} \frac{R_{kl}}{(s+b_k)}$$

$i = 1, 2, \cdots, p$

$j = 1, 2, \cdots, n$

$p \leq (n+1)$

Defining

$$R_{k\ell} = \left. \frac{(s+b_k) \prod_i (s+a_i)}{(s+b_0)^{m-\ell+1} \prod_j (s+b_j)} \right|_{s=-b_k}$$

The values of R_{kl} can be determined from this equation and the C's can be found from:

$$C_m = \left. \frac{\prod_i (s+a_i)}{\prod_j (s+b_j)} \right|_{s=-b_0}$$

$$C_\ell = \delta_{(m+n-\ell)(p)} - \sum_{k=1}^{n} R_{k\ell}$$

$$\delta_{(m+n-\ell)(p)} = \begin{cases} 1 \text{ for } & m+n-\ell = p \\ 0 \text{ for } & m+n-\ell \neq p \end{cases}$$

for values of ℓ such that

$$1 < \ell \leq m-1$$

E. TWO SECOND ORDER POLES:

More than one high order pole can be handled also:

$$\frac{1}{(s+b_1)^2 (s+b_2)^2} = \frac{A}{(s+b_1)^2} + \frac{B}{(s+b_1)} + \frac{C}{(s+b_2)^2} + \frac{D}{(s+b_2)}$$

A and C are easily found. They are:

$$A = \frac{1}{(b_2-b_1)^2} \qquad C = \frac{1}{(b_1-b_2)^2}$$

Then multiplying both sides of the original equation by $(s+b_1)$ gives:

$$\frac{1}{(s+b_1)(s+b_2)^2} = \frac{A}{(s+b_1)} + B + \frac{C(s+b_1)}{(s+b_2)^2} + \frac{D(s+b_1)}{(s+b_2)}$$

The last two terms can be expanded by the same techniques to give four terms, the first two are:

$$C\left[\frac{(b_1-b_2)}{(s+b_2)^2} + \frac{1}{(s+b_2)}\right]$$

The coefficient of one for the $(s+b_2)$ term results because the numerator is one order less than the denominator. Therefore, the sum of the residues is one. There is only one first order pole term in the expansion so its coefficient is this residue of one.

The last term expands into

$$\frac{D(b_1-b_2)}{(s+b_2)} + D$$

This gives

$$\frac{1}{(s+b_1)(s+b_2)^2} = \frac{A}{(s+b_1)} + B + \frac{C(b_1-b_2)}{(s+b_2)^2} + \frac{C}{(s+b_2)} + \frac{D(b_1-b_2)}{(s+b_2)} + D$$

the coefficient of the second order term can be more easily evaluated directly. Calling this coefficient E gives

$$C(b_1-b_2) \equiv E = \left.\frac{1\,\cancel{(s+b_2)^2}}{(s+b_1)\,\cancel{(s+b_2)^2}}\right|_{s=-b_2}$$

$$E = \frac{1}{(b_1-b_2)}$$

The sum of the residues is zero so

$$A + C + D(b_1 - b_2) = 0$$

Therefore

$$D = -\frac{A + C}{(b_1 - b_2)}$$

But

$$\frac{1}{(b_1 - b_2)} = E$$

So

$$D = -(A + C)E = -2E^3$$

And the last coefficient is found to be

$$B = -D$$

Thus all the coefficients have been found by algebra, no differentiation was needed. The resulting equation can be reduced to

$$\frac{1}{(s+b_1)^2(s+b_2)^2} = E^2\left[\frac{1}{(s+b_1)^2} + \frac{E}{(s+b_1)} + \frac{1}{(s+b_2)^2} - \frac{E}{(s+b_2)}\right]$$

Where

$$E = \frac{1}{(b_1 - b_2)}$$

These operations can be extended to higher order poles as required. With the basic rule about the sum of the residues, applied with usual algebraic manipulations, it appears that all differentiation can be avoided.

III ADDITIONAL PROPERTIES OF RESIDUES

A. For a normalized polynomial with the same order numerator and denominator

$$P = \frac{s^n + a_1 s^{n-1} + \cdots + a_n}{s^n + b_1 s^{n-1} + \cdots + b_n}$$

the sum of the residues is $a_1 - b_1$

The proof can be obtained by observing that $\frac{P-1}{a_1 - b_1}$ is a normalized polynomial with a numerator of one degree less than the denominator. The sum of the residues must be one. $P-1$ has the same residues as P and they must total $(a_1 - b_1)$.

B. If the numerator and denominator of a polynomial P are of the same order then the sum of the residues of $P+\frac{1}{P}$ will be zero. By (A)

$$\Sigma R(P) = (a_1 - b_1)$$

and

$$\Sigma R\left(\frac{1}{P}\right) = (b_1 - a_1)$$

then

$$\Sigma R(P) + \Sigma R\left(\frac{1}{P}\right) = 0$$

C. If the numerator of a polynomial is one degree higher than the denominator

$$P = \frac{s^{n+1} + a_1 s^n + a_2 s^{n-1} + \cdots + a_{n+1}}{s^n + b_1 s^{n-1} + b_2 s^{n-R} + \cdots + b_n}$$

then the

$$\Sigma R(P) = (a_2 - b_2) - b_1(a_1 - b_1)$$

This can be shown in a manner similar to A.

D. For a polynomial in which the numerator is $(n-1)$ larger than the denominator the first n coefficients of the numerator and the first n coefficients of the denominator are sufficient to determine the sum of the residues.

CONCLUSION

A theorem concerning the sum of the residues of the partial fraction expansion of a ratio of polynomials has been presented. It has been applied to the practical determinization of the residues of several often-encountered expressions. In addition, other relations have been given which may prove useful in expansions of greater complexity.

APPENDIX

Theorem: For a normalized ratio of polynomials:

A. If the denominator is one degree higher than the numerator, the sum of the residues is one.

B. If the denominator is two or more degrees higher than the numerator, the sum of the residues is zero.

Proof: Let the polynomial ratio be written in the form:

$$P = \frac{s^{n-r} + a_1 s^{n-(r+1)} + \cdots + a_{(r+1)}}{s^n + b_1 s^{n-1} + \cdots + b_n}$$

Let s be written in the form:

$$s = Re^{j\theta}$$

Then for fixed R

$$ds = jRe^{j\theta} d\theta$$

All of the poles are near the origin so that as R approaches infinity the value of the polynomial is approaching

$$\lim_{R \to \infty} P \to s^{-r} = \frac{1}{(Re^{j\theta})^r}$$

and the closed loop integral

$$\lim_{R \to \infty} \oint P\,ds \to \lim_{R \to \infty} \frac{j}{R^{(r-1)}} \int_0^{2\pi} \frac{d\theta}{e^{j\theta(r-1)}} = \begin{cases} 2\pi j & \text{for } r = 1 \\ 0 & \text{for } r \geqslant 2 \end{cases}$$

therefore according to the residue theorem the sum of the residues will be

$$\Sigma R_k = \begin{cases} 1 & \text{for } r = 1 \\ 0 & \text{for } r \geqslant 2 \end{cases}$$